

# ON BIRNBAUM–SAUNDERS INFERENCE

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**ABSTRACT.** The Birnbaum–Saunders distribution, also known as the fatigue-life distribution, is frequently used in reliability studies. We obtain adjustments to the Birnbaum–Saunders profile likelihood function. The modified versions of the likelihood function were obtained for both the shape and scale parameters, i.e., we take the shape parameter to be of interest and the scale parameter to be of nuisance, and then consider the situation in which the interest lies in performing inference on the scale parameter with the shape parameter entering the modeling in nuisance fashion. Modified profile maximum likelihood estimators are obtained by maximizing the corresponding adjusted likelihood functions. We present numerical evidence on the finite sample behavior of the different estimators and associated likelihood ratio tests. The results favor the adjusted estimators and tests we propose. A novel aspect of the profile likelihood adjustments obtained in this paper is that they yield improved point estimators *and* tests. The two profile likelihood adjustments work well when inference is made on the shape parameter, and one of them displays superior behavior when it comes to performing hypothesis testing inference on the scale parameter. Two empirical applications are briefly presented.

## 1. INTRODUCTION

Birnbaum and Saunders (1969a) derived a two-parameter distribution using a set-up in which failure time due to fatigue under cyclic loading when failure follows from the development and growth of a dominant crack. According to Marshall and Olkin (2007), the Birnbaum–Saunders distribution has appeared in several different contexts, and various derivations of the distribution are known. According to them (pp. 466–467), “it was given by Fletcher (1911), and according to Schrödinger (1915) it was obtained by Konstantinowsky (1914);” additionally, “it was obtained by Freudental and Shinozuka (1961), but it was the derivation of Birnbaum and Saunders (1969a) that brought the usefulness of the distribution into clear focus.” Desmond (1985) derived the same distribution in a more general setting; he used a biological model and relaxed several of the assumptions made by the original authors. The relationship between the the Birnbaum–Saunders and inverse Gaussian distributions was explored by Desmond (1986). It can shown that

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the Birnbaum–Saunders distribution is a mixture between an inverse Gaussian distribution and a generalized inverse Gaussian distribution; see Bhattacharyya and Fries (1982).

The random variable  $T$  is said to be Birnbaum–Saunders distributed, denoted  $T \sim \mathcal{BS}(\alpha, \beta)$ , if its density function is given by

$$f(t; \alpha, \beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right],$$

$t, \alpha, \beta > 0$ , where  $\alpha$  is the shape parameter and  $\beta$  is the scale parameter. It is noteworthy that the reciprocal property holds for the Birnbaum–Saunders distribution:  $T^{-1} \sim \mathcal{BS}(\alpha, \beta^{-1})$ ; see Saunders (1974). It is easy to show that  $\mathbb{E}(T) = \beta \left( 1 + \frac{1}{2}\alpha^2 \right)$ ,  $\text{var}(T) = (\alpha\beta)^2 \left( 1 + \frac{5}{4}\alpha^2 \right)$ ,  $\mathbb{E}(T^{-1}) = \beta^{-1} \left( 1 + \frac{1}{2}\alpha^2 \right)$  and  $\text{var}(T^{-1}) = \alpha^2 \beta^{-2} \left( 1 + \frac{5}{4}\alpha^2 \right)$ .

The Birnbaum–Saunders distribution function is

$$F(t; \alpha, \beta) = \Phi \left( \frac{1}{\alpha} \left[ \left( \frac{t}{\beta} \right)^{1/2} - \left( \frac{\beta}{t} \right)^{1/2} \right] \right), \quad 0 < t < \infty, \quad \alpha, \beta > 0,$$

where  $\Phi(\cdot)$  denotes the standard normal distribution function. Note that  $\beta$  is the median of the distribution:  $F_T(\beta) = \Phi(0) = 0.5$ . It was shown by Kundu, Kannan and Balakrishnan (2008) that the Birnbaum–Saunders hazard function is an upside down function for all values of the shape and scale parameters. Hence, the distribution is useful in a number of practical situations where the hazard function increases up to a point and then decreases. The authors have also addressed the important issue of performing inference on the point at which the hazard function reaches its maximum.

Oftentimes the interest lies in performing inference on a subset of the parameters that index the model; such parameters are said to be of *interest*, and the remaining ones are *nuisance parameters*. For instance, in Birnbaum–Saunders reliability studies, one is typically interested in performing inference on one of the parameters that index the model, the other parameter entering the modeling process in nuisance fashion. In the presence of nuisance parameters, inferences are usually based on the profile likelihood function, which is obtained by replacing, in the likelihood function, the nuisance parameters by their corresponding maximum likelihood estimators for fixed values of the parameters of interest. The resulting function — the profile likelihood function — will only depend on the parameters of interest. It is noteworthy, however, that such a function is not a true likelihood function, and some of the properties that hold for likelihood functions are no longer valid; in particular, there exist score and information biases that do not vanish as the sample size increases. Several adjustments to the profile likelihood function have been proposed; see, e.g., Barndorff–Nielsen (1983, 1994), Cox and Reid (1987, 1992), McCullagh and Tibshirani (1990) and Stern (1997). The main idea behind these adjustments is to add a term to the log-likelihood function prior to maximizing it, in order to overcome the aforementioned shortcomings.

In this paper we shall use the results in Barndorff–Nielsen (1983), Severini (1998, 1999) and Cox and Reid (1987) to obtain adjustments to the Birnbaum–Saunders profile likelihood function. A novel aspect of this approach is that the effect of the nuisance parameter on the inference performed on the interest parameter is greatly reduced. It is also noteworthy that likelihood ratio type tests constructed using the adjusted profile likelihood function typically have superior finite sample performance. In short, by adjusting the profile likelihood function and then maximizing it one can perform reliable point estimation and hypothesis testing inference even when the sample size is small. Our results shall allow practitioners to perform reliable inference when using the Birnbaum–Saunders model in small samples. A motivation for our analysis lies in the important situation in which one wishes to make inferences on the the median failure time in a reliability study. As we have seen, the median of the Birnbaum–Saunders distribution is  $\beta$ , one of the parameters that index such a distribution. Therefore, here  $\alpha$  is a nuisance parameter. There are also situations where the interest lies in performing statistical inference on the shape parameter  $\alpha$ , with  $\beta$  figuring as a nuisance unknown quantity. It is thus important to develop reliable and accurate inference strategies that are not sensitive (or, at least, less sensitive) to the parameter that enters the modeling in nuisance fashion. This is our chief goal.

The paper unfolds as follows. Section 2 introduces adjustments to the profile likelihood function when the interest lies in performing inference in the presence of nuisance parameters. In Section 3, we derive adjustments to the Birnbaum–Saunders profile likelihood function. The use of such adjustments delivers, as noted above, improved estimation *and* testing inference in small samples. Alternative inference strategies are presented in Section 4. Numerical results are presented in Sections 5 and 6, and two applications are presented in Section 7. Finally, Section 8 summarizes the main findings and lists directions for future research.

## 2. PROFILE LIKELIHOOD FUNCTION AND ADJUSTMENTS

Let  $t_1, \dots, t_n$  be independent and identically distributed random variables with joint density  $f(t; \theta)$ , where  $\theta \subseteq \mathbb{R}^p$  is a  $p$ -vector of unknown parameters and  $\mathbf{t} = (t_1, \dots, t_n)^\top$ . In what follows, we shall partition  $\theta$  as  $\theta = (\tau^\top, \phi^\top)^\top$ , where  $\tau$ , a  $q$ -vector, contains the parameters of interest and  $\phi$ , a  $(p - q)$ -vector, contains the nuisance parameters.

Inference can be based on the profile likelihood function, defined as  $L_p(\tau) = L(\tau, \widehat{\phi}_\tau)$ , where  $L(\cdot)$  is the usual likelihood function and  $\widehat{\phi}_\tau$  is the restricted maximum likelihood estimator of  $\phi$  given  $\tau$ . The profile likelihood is not a true likelihood, and some of the properties that hold for a genuine likelihood do not hold for its profiled version. In particular, there exist score and information biases, both of order  $O(1)$ .

The interest lies in testing the null hypothesis  $\mathcal{H}_0 : \tau = \tau_0$  against  $\mathcal{H}_1 : \tau \neq \tau_0$ , where  $\tau_0$  is a given  $q$ -vector of scalars. The likelihood ratio statistic obtained from the profile likelihood function is

$$LR = 2 \left\{ \ell(\widehat{\tau}, \widehat{\phi}) - \ell(\tau, \widehat{\phi}_\tau) \right\} = 2 \left\{ \ell_p(\widehat{\tau}) - \ell_p(\tau) \right\}.$$

Here,  $\widehat{\tau}$  and  $\widehat{\phi}$  are the maximum likelihood estimators of  $\tau$  and  $\phi$ , respectively,  $\ell(\cdot)$  is the log-likelihood function and  $\ell_p(\cdot)$  is the profile log-likelihood function. Under the null hypothesis,  $LR \rightsquigarrow \chi_q^2$ , where  $\rightsquigarrow$  denotes convergence in distribution.

Several adjustments to the profile likelihood function have been proposed in the literature; see, e.g., Severini (2000, Chapter 9), Pace and Salvan (1997, Chapter 11) and the referenced therein for details.

Barndorff–Nielsen (1983) proposed an adjusted profile likelihood function which is invariant under reparameterizations of the form  $(\tau, \phi) \rightarrow (\tau, \zeta(\tau, \phi))$ , where  $\tau$  is the vector of parameters of interest,  $\phi$  is the vector of nuisance parameters and  $\zeta$  is a function of  $\tau$  and  $\phi$ . His proposal follows from the  $p^*$  formula, which is an approximation to the conditional density of the maximum likelihood estimator given an ancillary statistic. The proposed adjusted profile likelihood function is

$$L_{BN}(\tau) = \left| \frac{\partial \widehat{\phi}_\tau}{\partial \widehat{\phi}} \right|^{-1} |j_{\phi\phi}(\tau, \widehat{\phi}_\tau)|^{-1/2} L_p(\tau),$$

where  $\partial \widehat{\phi}_\tau / \partial \widehat{\phi}$  is the matrix of partial derivatives of  $\widehat{\phi}_\tau$  with respect to  $\widehat{\phi}$ ,  $j_{\phi\phi}(\tau, \phi) = -\partial^2 \ell / \partial \phi \partial \phi^\top$  is the observed information matrix for  $\phi$  when  $\tau$  is fixed and  $L_p(\tau)$  is the profile likelihood function for  $\tau$ .

There is an alternative expression for  $L_{BN}$  that does not involve  $|\partial \widehat{\phi}_\tau / \partial \widehat{\phi}|$ ; it involves, nonetheless, a sample space derivative and requires an ancillary statistic  $a$  such that  $(\tau, \widehat{\phi}, a)$  is minimal sufficient. It can be shown that

$$\frac{\partial \widehat{\phi}_\tau}{\partial \widehat{\phi}} = j_{\phi\phi}(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a)^{-1} \ell_{\phi; \widehat{\phi}}(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a),$$

where

$$\ell_{\phi; \widehat{\phi}}(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a) = \frac{\partial}{\partial \widehat{\phi}} \left( \frac{\partial \ell(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a)}{\partial \phi} \right).$$

Here,  $\ell_{\phi; \widehat{\phi}}(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a)$  and  $j_{\phi\phi}(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi}, a)$  are the log-likelihood function and the observed information for  $\phi$ , respectively. They depend on the data only through the minimal sufficient statistic.

Some approximations to the sample space derivative of the log-likelihood function have been proposed. Severini (1998) obtained an approximation to Barndorff–Nielsen’s adjusted profile likelihood function that requires neither a sample space derivative nor an ancillary statistic. It is given by

$$\bar{\ell}_{BN}(\tau) = \ell_p(\tau) + \frac{1}{2} \log |j_{\phi\phi}(\widehat{\tau}, \widehat{\phi}_\tau)| - \log |I_\phi(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi})|,$$

where

$$I_\phi(\tau, \phi; \tau_0, \phi_0) = \mathbb{E}_{(\tau_0, \phi_0)} \left\{ \ell_\phi(\tau, \phi) \ell_\phi(\tau_0, \phi_0)^\top \right\} \quad (1)$$

is the covariance matrix of log-likelihood derivatives and  $\ell_\phi(\tau, \phi) = \partial \ell / \partial \phi$ . The approximation error is of order  $O(n^{-1/2})$ . The corresponding maximum likelihood estimator shall be denoted as  $\widehat{\tau}_{BN}$ .

An alternative approximation, with the same approximation error, was proposed by Severini (1999):

$$\check{\ell}_{BN}(\tau) = \ell_p(\tau) + \frac{1}{2} \log |j_{\phi\phi}(\widehat{\tau}, \widehat{\phi}_\tau)| - \log |\check{I}_\phi(\tau, \widehat{\phi}_\tau; \widehat{\tau}, \widehat{\phi})|,$$

where

$$\check{I}_\phi(\tau, \phi; \tau_0, \phi_0) = \sum_{j=1}^n \ell_\phi^{(j)}(\tau, \phi) \ell_\phi^{(j)}(\tau_0, \phi_0)^\top, \quad (2)$$

$\ell_\theta^{(j)}(\theta) = (\ell_\tau^{(j)}(\theta), \ell_\phi^{(j)}(\theta))$  being the score function for the  $j$ th observation. This approximation can be easily computed and is particularly useful in situations where one is not able to compute expected values of log-likelihood derivatives. The corresponding maximum likelihood estimator shall be denoted as  $\widehat{\tau}_{BN}$ .

Cox and Reid (1987) defined an adjusted profile likelihood function, where an adjustment term is included into the likelihood function prior to maximization. It approximates the conditional density function of the observations given the nuisance parameter maximum likelihood estimator and can be written as

$$L_{CR}(\tau) = |j_{\phi\phi}(\tau, \widehat{\phi}_\tau)|^{-1/2} L_p(\tau).$$

Taking logs we obtain

$$\ell_{CR}(\tau) = \ell(\tau, \widehat{\phi}_\tau) - \frac{1}{2} \log |j_{\phi\phi}(\tau, \widehat{\phi}_\tau)|. \quad (3)$$

Note that this function is the penalized counterpart of the log-likelihood function, the penalty term taking into account information on the nuisance parameter. The maximizer of  $\ell_{CR}(\tau)$  shall be denoted as  $\widehat{\tau}_{CR}$ . It is noteworthy that the score bias is of order  $O(n^{-1})$ , but the information bias remains  $O(1)$ .

The derivation of  $\ell_{CR}(\tau)$  requires that  $\tau$  and  $\phi$  be orthogonal, i.e., that the elements of the score vector,  $\partial\ell/\partial\tau$  and  $\partial\ell/\partial\phi$ , be uncorrelated which implies that  $i_{\tau\phi} = 0$ . When  $i_{\tau\phi} \neq 0$ , it is necessary to find a reparameterization of the form  $(\tau, \lambda(\tau, \phi))$ , where  $\tau$  and  $\lambda$  are orthogonal. It is noteworthy that such a reparameterization cannot always be found, except when the parameter of interest is scalar. We also note that the Cox and Reid adjustment is not invariant under reparameterizations of the form  $(\tau, \phi) \rightarrow (\tau, \zeta(\tau, \phi))$ , unlike Barndorff–Nielsen's adjustment.

### 3. THE BIRNBAUM–SAUNDERS ADJUSTED PROFILE LIKELIHOODS

At the outset, let  $\alpha$  be the parameter of interest and  $\beta$  the nuisance parameter. Also, let  $\mathbf{t} = (t_1, \dots, t_n)^\top$  denote a random sample of size  $n$  from the Birnbaum–Saunders distribution. The log-likelihood function is

$$\ell(\alpha, \beta) = -n \log(\alpha\beta) + \sum_{i=1}^n \log \left[ \left( \frac{\beta}{t_i} \right)^{1/2} + \left( \frac{\beta}{t_i} \right)^{3/2} \right] - \frac{1}{2\alpha^2} \sum_{i=1}^n \left( \frac{t_i}{\beta} + \frac{\beta}{t_i} - 2 \right).$$

For fixed  $\alpha$ , the restricted maximum likelihood estimator of  $\beta$ ,  $\widehat{\beta}_\alpha$ , is the positive root of the following nonlinear equation:

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0,$$

where

$$s = \frac{1}{n} \sum_{i=1}^n t_i, \quad r = \left( \frac{1}{n} \sum_{i=1}^n t_i^{-1} \right)^{-1} \quad \text{and} \quad K(\beta) = \left[ \frac{1}{n} \sum_{i=1}^n (\beta + t_i)^{-1} \right]^{-1}.$$

Note that  $\widehat{\beta}_\alpha$  does not have a closed-form expression, and, as a result, it must be obtained using restricted nonlinear optimization methods; see, e.g., Nocedal and Wright (1999, Chapter 18). (We note that the maximum likelihood estimator of  $\beta$  for fixed  $\alpha$  equals the maximum likelihood estimator of  $\beta$ , that is,  $\widehat{\beta}_\alpha = \widehat{\beta}$ .) By replacing  $\beta$  by  $\widehat{\beta}_\alpha$  in  $\ell(\alpha, \beta)$  we obtain the profile log-likelihood function given by

$$\begin{aligned} \ell_p(\alpha) &= -n \log \alpha - n \log \widehat{\beta}_\alpha + \sum_{i=1}^n \log \left[ \left( \frac{\widehat{\beta}_\alpha}{t_i} \right)^{1/2} + \left( \frac{\widehat{\beta}_\alpha}{t_i} \right)^{3/2} \right] \\ &\quad - \frac{n}{2\alpha^2} \left( \frac{r}{\widehat{\beta}_\alpha} + \frac{\widehat{\beta}_\alpha}{s} - 2 \right). \end{aligned}$$

The asymptotic distribution of the vector of maximum likelihood estimators of the parameters that index the Birnbaum–Saunders distribution was obtained by Englehardt, Bain and Wright (1981). A simpler expression for Fisher's information matrix was obtained by Lemonte, Cribari–Neto and Vasconcellos (2007).

In what follows, we shall obtain the adjusted profile likelihoods described in Section 2. Note that the interest and nuisance parameters are orthogonal. The adjusted profile log-likelihood function of Cox and Reid (1987) for  $\alpha$  can be expressed as

$$\ell_{CR}(\alpha) = \ell_p(\alpha) - \frac{1}{2} \log |j_{\beta\beta}(\alpha, \widehat{\beta}_\alpha)|,$$

where

$$j_{\beta\beta}(\alpha, \widehat{\beta}_\alpha) = -\frac{n}{\widehat{\beta}_\alpha^2} + \frac{n}{2} \left( \frac{1}{\widehat{\beta}_\alpha^2} + \frac{2K'(\widehat{\beta}_\alpha)}{K^2(\widehat{\beta}_\alpha)} \right) + \frac{n}{\alpha^2} \frac{r}{\widehat{\beta}_\alpha^3}$$

and

$$K'(\beta) = \frac{n \sum_{i=1}^n (\beta + t_i)^{-2}}{\left[ \sum_{i=1}^n (\beta + t_i)^{-1} \right]^2}.$$

$\widehat{\alpha}_{CR}$  is the adjusted profile maximum likelihood estimator of  $\alpha$ ; it does not have closed-form and must be obtained numerically.

The Barndorff–Nielsen (1983) adjusted profile log-likelihood function for  $\alpha$  can be written as

$$\ell_{BN}(\alpha) = \ell_p(\alpha) + \log \frac{|j_{\beta\beta}(\alpha, \widehat{\beta}_\alpha)|^{1/2}}{|j_{\beta;\beta}(\alpha, \widehat{\beta}_\alpha)|}.$$

Instead of obtaining the term  $j_{\beta;\widehat{\beta}}(\alpha, \widehat{\beta}_\alpha)$  in  $\ell_{BN}$ , we shall obtain  $\check{I}(\alpha, \widehat{\beta}_\alpha; \widehat{\alpha}, \widehat{\beta})$  given in (2) using, thus, Severini's (1999) approximation. After some algebra, we obtain

$$\begin{aligned} \check{I}(\alpha, \widehat{\beta}_\alpha; \widehat{\alpha}, \widehat{\beta}) &= \frac{n}{\widehat{\beta}^2} - \frac{1}{\widehat{\beta}} \sum_{j=1}^n A_j - \frac{1}{4} \left[ \sum_{j=1}^n A_j^2 + \frac{1}{\alpha^2 \widehat{\alpha}^2} \sum_{j=1}^n B_j^2 \right] \\ &\quad - \frac{1}{4} \left[ \left( \frac{1}{\alpha^2} + \frac{1}{\widehat{\alpha}^2} \right) \left( \sum_{j=1}^n A_j B_j - \frac{2}{\widehat{\beta}} \sum_{j=1}^n B_j \right) \right], \end{aligned}$$

where

$$A_j = \left( \frac{t_j^{-1/2} \widehat{\beta}^{-1/2} + 3 \widehat{\beta}^{1/2} t_j^{-3/2}}{t_j^{-1/2} \widehat{\beta}^{1/2} + \widehat{\beta}^{3/2} t_j^{-3/2}} \right) \quad \text{and} \quad B_j = \left( \frac{t_j}{\widehat{\beta}^2} - \frac{1}{t_j} \right).$$

The adjusted profile maximum likelihood estimator  $\widehat{\alpha}_{BN}$  of  $\alpha$  cannot be expressed in closed-form; it has to be computed by numerically maximizing the associated log-likelihood function.

The likelihood ratio test statistics obtained from the adjusted profile log-likelihood functions given above for the test of  $\mathcal{H}_0 : \alpha = \alpha_0$  against  $\mathcal{H}_1 : \alpha \neq \alpha_0$  are

$$LR_{CR}(\alpha) = 2 \{ \ell_{CR}(\widehat{\alpha}_{CR}) - \ell_{CR}(\alpha_0) \}$$

and

$$LR_{BN}(\alpha) = 2 \{ \ell_{BN}(\widehat{\alpha}_{BN}) - \ell_{BN}(\alpha_0) \},$$

where  $\widehat{\alpha}_{CR}$  and  $\widehat{\alpha}_{BN}$  are the values of  $\alpha$  that maximize  $\ell_{CR}(\alpha)$  and  $\ell_{BN}(\alpha)$ , respectively. These test statistics are asymptotically distributed as  $\chi_1^2$  under the null hypothesis.

We shall now consider  $\beta$  as the parameter of interest and view  $\alpha$  as a nuisance parameter. For fixed  $\beta$ , we write the restricted maximum likelihood estimator of  $\alpha$  as

$$\widehat{\alpha}_\beta = \left( \frac{r}{\beta} + \frac{\beta}{s} - 2 \right)^{1/2}.$$

By plugging  $\widehat{\alpha}_\beta$  into the log-likelihood function we obtain the following profile log-likelihood function:

$$\begin{aligned} \ell_p(\beta) &= \ell(\widehat{\alpha}_\beta, \beta) = -\frac{n}{2} \log \left( \frac{r}{\beta} + \frac{\beta}{s} - 2 \right) - n \log \beta \\ &\quad + \sum_{i=1}^n \log \left[ \left( \frac{\beta}{t_i} \right)^{1/2} + \left( \frac{\beta}{t_i} \right)^{3/2} \right]. \end{aligned}$$

The  $j_{\alpha\alpha}(\alpha, \beta)$  block of the observed information matrix evaluated at  $(\widehat{\alpha}_\beta, \beta)$  can be written as

$$j_{\alpha\alpha}(\widehat{\alpha}_\beta, \beta) = -2n \left( \frac{r}{\beta} + \frac{\beta}{s} - 2 \right)^{-1}.$$

From (3) it follows that Cox and Reid's adjusted profile log-likelihood function for  $\beta$  is

$$\ell_{CR}(\beta) = \ell_p(\beta) + \frac{1}{2} \log \left| \frac{r}{\beta} + \frac{\beta}{s} - 2 \right|.$$

The estimator  $\widehat{\beta}_{CR}$ , like the previous estimators, does not have closed-form.

The Barndorff–Nielsen adjusted profile log-likelihood function can be expressed as

$$\ell_{BN}(\beta) = \ell_p(\beta) + \log \frac{|j_{\alpha\alpha}(\widehat{\alpha}_\beta, \beta)|^{1/2}}{|\ell_{\alpha, \widehat{\alpha}}(\widehat{\alpha}_\beta, \beta)|}.$$

We use Severini's (1998) approximation and replace  $\ell_{\alpha, \widehat{\alpha}}(\widehat{\alpha}_\beta, \beta)$ , in  $\ell_{BN}(\beta)$ , by  $I(\widehat{\alpha}_\beta, \beta; \widehat{\alpha}, \widehat{\beta})$  given in (1). We arrive at

$$I(\widehat{\alpha}_\beta, \beta; \widehat{\alpha}, \widehat{\beta}) = \frac{n\widehat{\alpha}}{\widehat{\alpha}_\beta^3} \left( \frac{\widehat{\beta}}{\beta} + \frac{\beta}{\widehat{\beta}} \right).$$

The corresponding estimator,  $\widehat{\beta}_{BN}$ , does not have closed-form.

The likelihood ratio test statistics obtained from the above adjusted profile log-likelihood functions for the test of  $\mathcal{H}_0 : \beta = \beta_0$  against  $\mathcal{H}_1 : \beta \neq \beta_0$  are

$$LR_{CR}(\beta) = 2 \left\{ \ell_{CR}(\widehat{\beta}_{CR}) - \ell_{CR}(\beta_0) \right\}$$

and

$$LR_{BN}(\beta) = 2 \left\{ \ell_{BN}(\widehat{\beta}_{BN}) - \ell_{BN}(\beta_0) \right\},$$

where  $\widehat{\beta}_{CR}$  and  $\widehat{\beta}_{BN}$  are the values of  $\beta$  that maximize  $\ell_{CR}(\beta)$  and  $\ell_{BN}(\beta)$ , respectively. The two test statistics are asymptotically distributed as  $\chi_1^2$  under  $\mathcal{H}_0$ .

#### 4. ALTERNATIVE INFERENCE STRATEGIES

Some alternative point estimators for the parameters that index the Birnbaum–Saunders distributions have been proposed in the literature. Ng, Kundu and Balakrishnan (2003) obtained modified moment estimators for  $\alpha$  and  $\beta$ . As before, let  $s = \bar{t} = n^{-1} \sum_{i=1}^n t_i$  (sample arithmetic mean) and  $r = \left( n^{-1} \sum_{i=1}^n t_i^{-1} \right)^{-1}$  (sample harmonic mean). The estimators can then be written as

$$\bar{\alpha}_{Ng} = \sqrt{s \left( \sqrt{\frac{s}{r}} - 1 \right)} \quad \text{and} \quad \bar{\beta}_{Ng} = \sqrt{sr}.$$

Ng, Kundu and Balakrishnan (2003) have also proposed jackknife estimators for  $\alpha$  and  $\beta$ . The underlying idea is to remove observation  $t_j$  from the random sample  $\mathbf{t} = (t_1, t_2, \dots, t_n)^\top$ , and to estimate the parameters based on the remaining  $n - 1$  observations; this is done for  $j = 1, \dots, n$ . We shall denote the jackknife maximum likelihood estimators as  $\bar{\alpha}_{NgJMME}$  and  $\bar{\beta}_{NgJMME}$ ; the jackknife moment estimator are  $\bar{\alpha}_{NgJMME}$  and  $\bar{\beta}_{NgJMME}$ .



From and Li (2006) also proposed alternative estimators for the two parameters that index the Birnbaum–Saunders distribution. For instance, they proposed using

$$\check{\beta}_{F1} = \frac{\sum_{i=1}^n t_i^{1/2}}{\sum_{i=1}^n t_i^{-1/2}} \quad \text{and} \quad \check{\alpha}_{F1} = \sqrt{\frac{s}{\check{\beta}_{F1}} + \frac{\check{\beta}_{F1}}{r}} - 2.$$

The authors have also proposed a second estimator for  $(\alpha, \beta)$ . Their proposal is to estimate  $\beta$  using  $\check{\beta}_{F2} = \text{median}(t_1, \dots, t_n)$  since  $\beta$  equals the median of the Birnbaum–Saunders distribution. The corresponding estimator for  $\alpha$  is

$$\check{\alpha}_{F2} = \sqrt{\frac{-2 + 2\sqrt{1 + 5v}}{5}},$$

where  $v = \widehat{\sigma}^2 / \check{\beta}_{F2}$ ,  $\widehat{\sigma}^2$  being the sample variance, i.e.,  $\widehat{\sigma}^2 = (n-1)^{-1} \sum_{i=1}^n (t_i - \bar{t})^2$ .

Let  $t_{(1)}, \dots, t_{(n)}$  denote the order statistics of the sample  $t_1, \dots, t_n$ . The estimator for  $\beta$  is, as above, the sample median. For each  $t_{(i)}$ , solve

$$F(t_{(i)}; \alpha, \check{\beta}_{F2}) = \frac{i}{n+1}, \quad i = 1, \dots, n.$$

Let  $\widehat{\alpha}(i)$ ,  $i = 1, \dots, n$ , denote the solutions, where

$$\widehat{\alpha}(i) = \frac{h\left(\frac{t_{(i)}}{\check{\beta}_{F2}}\right)}{\Phi^{-1}\left(\frac{i}{n+1}\right)},$$

with  $h(t) = t^{1/2} - t^{-1/2}$ . The estimator is  $\check{\alpha}_{F3} = \text{median}(\widehat{\alpha}(1), \dots, \widehat{\alpha}(n))$ .

From and Li (2006) have also proposed yet another estimator for  $(\alpha, \beta)$ . Let  $0 < \lambda < 0.5$ , and let  $n_1 = n\lambda + 1$  and  $n_2 = n(1 - \lambda)$ , to the nearest integer. The proposed estimators are

$$\check{\beta}_{F4} = \frac{\sum_{i=n_1}^{n_2} t_{(i)}}{\sum_{i=n_1}^{n_2} \frac{1}{\sqrt{t_{(i)}}}} \quad \text{and} \quad \check{\alpha}_{F4} = \sqrt{\frac{\sum_{i=n_1}^{n_2} h^2\left(\frac{t_{(i)}}{\check{\beta}_{F2}}\right)}{\sum_{i=n_1}^{n_2} \left[\Phi^{-1}\left(\frac{i}{n+1}\right)\right]}}.$$

The authors suggest using  $\lambda = 0.05$ , so that only the middle 90% of order statistics are used.

An alternative hypothesis test was proposed by Lemonte, Cribari–Neto and Vasconcellos (2007). They derived a Bartlett-correction factor to the likelihood ratio statistic and obtained an approximate test whose error vanishes at a faster rate as the sample size increases. Let  $LR^*$  denote their test statistic. It follows that whereas  $\Pr(LR \leq x) = \Pr(\chi_1^2 \leq x) + O(n^{-1})$ , the correction yields  $\Pr(LR^* \leq x) = \Pr(\chi_1^2 \leq x) + O(n^{-2})$ , a clear improvement. [See Cribari–Neto and Cordeiro (1996) for a detailed review of Bartlett corrections.] Consider the following null hypotheses: (i)  $\mathcal{H}_0 : \alpha = 0.1$ , (ii)  $\mathcal{H}_0 : \alpha = 0.25$ , (iii)  $\mathcal{H}_0 : \alpha = 0.5$ , (iv)  $\mathcal{H}_0 : \alpha = 0.75$ , (v)  $\mathcal{H}_0 : \alpha = 1.0$ , (vi)  $\mathcal{H}_0 : \alpha = 2.0$ . The corresponding Bartlett-corrected test statistics

are<sup>1</sup>

$$LR^* = \frac{LR}{1 + 4.3918/n}, LR^* = \frac{LR}{1 + 3.2537/n}, LR^* = \frac{LR}{1 + 3.0414/n},$$

$$LR^* = \frac{LR}{1 + 2.5924/n}, LR^* = \frac{LR}{1 + 2.0307/n} \text{ and } LR^* = \frac{LR}{1 - 0.0445/n}.$$

## 5. NUMERICAL EVIDENCE

We shall now present Monte Carlo simulation results on the finite sample behavior of inference based on profile and adjusted profile likelihoods. All simulation experiments entail 10,000 replications. The shape parameter assumed two values, namely  $\alpha = 0.5, 1.0$ , and the scale parameter was set at  $\beta = 1.0$ . The simulations were performed using the Ox matrix programming language (Doornik, 2006). Likelihood maximizations were performed using the quasi-Newton method known as BFGS with analytical first derivatives; see Nocedal and Wright (1999) for details on the BFGS method.

Point estimation is evaluated using the following measures: mean, bias, variance, mean squared error (MSE), relative bias (RB), asymmetry and kurtosis. Relative bias is defined as  $100 \times (\text{bias}/\text{true parameter value})$ . Hypothesis testing inference on the parameter of interest is described through the null rejection rates of the profile and adjusted profile likelihood ratio tests. Power simulations were also performed.

Table 1 contains simulation results for the case where  $\alpha$  is the parameter of interest. The sample size is  $n = 10$ . Note that the estimators  $\widehat{\alpha}_{CR}$  and  $\widehat{\alpha}_{BN}$  are less biased than  $\widehat{\alpha}_p$ . For instance, when  $\alpha = 0.5$  the relative biases of  $\widehat{\alpha}_p$ ,  $\widehat{\alpha}_{CR}$  and  $\widehat{\alpha}_{BN}$  are 7.50%, 2.16% and 2.13%, respectively. Nevertheless, bias reduction is achieved at the expense of greater variability. It is also noteworthy that the small sample behavior of the two adjusted profile maximum likelihood estimators are similar. We also note that the skewness and kurtosis of  $\widehat{\alpha}_p$  are slightly closer to their asymptotic counterparts than those of  $\widehat{\alpha}_{CR}$  and  $\widehat{\alpha}_{BN}$ . (When the parameter of interest is  $\beta$ , the estimators  $\widehat{\beta}_p$ ,  $\widehat{\beta}_{CR}$  and  $\widehat{\beta}_{BN}$  coincide, since maximization of the profile likelihood function is equivalent to that of  $\ell_{CR}(\beta)$  or  $\ell_{BN}(\beta)$ . As a consequence, the profile and adjusted profile maximum likelihood estimators also coincide.)

Table 2 presents the null rejection rates (%) of the different likelihood ratio tests, i.e., the tests based on the statistics  $LR$ ,  $LR_{CR}$ ,  $LR_{BN}$  and  $LR^*$ , for the test of  $\mathcal{H}_0 : \alpha = \alpha_0$  against  $\mathcal{H}_1 : \alpha \neq \alpha_0$ , where  $\alpha_0$  is a given scalar; here,  $\alpha$  is the parameter of interest and  $\beta$  is the nuisance parameter, whose value is set at  $\beta = 1.0$ . The values set at the null hypothesis are  $\alpha_0 = 0.5, 1.0, 2.0$  and the sample sizes are  $n = 10, 25, 50$ . All entries are percentages. The figures in Table 2 show that the adjusted profile likelihood ratio tests ( $LR_{CR}$  and  $LR_{BN}$ ) outperform the usual profile likelihood ratio test ( $LR$ ). For instance, at the 10% nominal level and when  $\alpha_0 = 0.5$ , the rejection rate of the latter was 13.57% whereas the rejection rates

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<sup>1</sup>Note that the values of the Bartlett correction factor we give correct those given by the authors for cases (i) and (ii).

TABLE 1. Point estimation of  $\alpha$ .

$\alpha = 0.5$							
estimator	mean	bias	variance	MSE	RB(%)	asymmetry	kurtosis
$\widehat{\alpha}_p$	0.4625	-0.0375	0.0121	0.0135	7.5074	0.2194	6.0405
$\widehat{\alpha}_{CR}$	0.4892	-0.0109	0.0136	0.0138	2.1695	0.2355	6.3629
$\widehat{\alpha}_{BN}$	0.4893	-0.0107	0.0137	0.0138	2.1385	0.2356	6.3648
$\alpha = 1.0$							
estimator	mean	bias	variance	MSE	RB(%)	asymmetry	kurtosis
$\widehat{\alpha}_p$	0.9160	-0.0840	0.0472	0.0543	8.4020	0.3715	11.9221
$\widehat{\alpha}_{CR}$	0.9752	-0.0248	0.0548	0.0554	2.4841	0.3732	12.6356
$\widehat{\alpha}_{BN}$	0.9792	-0.0208	0.0567	0.0572	2.0834	0.3733	12.6829

of  $LR_{CR}$  and  $LR_{BN}$  were 10.85% and 10.86%, respectively. The likelihood ratio test is clearly liberal, the null hypothesis being rejected more often than expected. The adjusted tests display much smaller size distortions than the likelihood ratio test. It is also noteworthy that the finite sample behavior of the likelihood ratio test deteriorates as the value of the shape parameter increases, especially when the sample size is small; the adjusted tests remain reliable. The Bartlett-corrected test is clearly outperformed by the adjusted profile likelihood tests when  $\alpha$  is large and  $n$  is small. For example, when  $\alpha = 2.0$ ,  $n = 10$  and the nominal level is 5%, the null rejection rate of the Bartlett-corrected test is 9.40% whereas the adjusted profile likelihood tests reject the null hypothesis 5.20% ( $LR_{CR}$ ) and 4.58% ( $LR_{BN}$ ) of the time. Note also that the tests based on  $LR_{CR}$  and  $LR_{BN}$  display similar small sample behavior, especially when the value of the shape parameter is small. (Values of  $\alpha$  between 0.1 and 0.5 are common in fatigue studies.)

We have also performed simulations under the alternative hypothesis. The powers of the tests  $LR$ ,  $LR_{CR}$ ,  $LR_{BN}$  and  $LR^*$  at the 5% and 1% nominal levels were computed for values of  $\alpha$  ranging from 0.12 to 0.28, the null hypothesis under test being  $\mathcal{H}_0 : \alpha = \alpha_0$ . The tests were carried out using exact critical values, which were estimated in the size simulations. This was done so that the different tests have the same size, and power comparisons become meaningful. The results are presented in Table 3 and were obtained using  $n = 10$ ,  $\alpha = 0.10$  and  $\beta = 1.0$ . (All entries are percentages.) We note that  $LR$  is slightly less powerful than  $LR_{BN}$  and  $LR_{CR}$ . For example, when  $\alpha = 0.20$  and at the 5% nominal level, the nonnull rejection rates of these tests were equal to 77.86%, 83.81% and 83.81%, respectively; the nonnull rejection rate of the Bartlett-corrected test was 77.87%.

Figure 1 plots the relative quantile discrepancies of the three test statistics against the corresponding asymptotic quantiles. Relative quantile discrepancy is defined as the difference between exact (estimated by simulation) and asymptotic quantiles divided by the latter. The closer to zero the relative quantile discrepancies, the better the approximation of the exact null distribution of the test statistic by the limiting  $\chi_1^2$  distribution. It is noteworthy that the relative quantile discrepancies of the adjusted test statistics are considerably closer to zero than those of the likelihood ratio test statistic, which oscillate around 18%. The relative quantile discrepancies of the two adjusted test statistics are very similar.

Figure 2 plots the relative size distortions against the corresponding nominal levels of the tests. Relative size distortion is defined as the difference between  $p$ -values (estimated by simulation) and nominal levels divided by the latter. Note that the relative size distortion of the likelihood ratio test increases rapidly as the nominal level of the test decreases, which does not occur for the adjusted tests. Note also that the relative size distortions of the two adjusted tests are very similar.

Table 4 contains the null rejection rates (again expressed as percentages) of the three tests ( $LR$ ,  $LR_{CR}$  and  $LR_{BN}$ ) for the test  $\mathcal{H}_0 : \beta = \beta_0$ . (Note that here  $\beta$  is the parameter of interest.) Since  $\beta$  functions as a multiplier, as explained earlier, we have only performed simulations using  $\beta = 1.0$ ; four different values of  $\alpha$  were used, namely: 0.1, 0.5, 1.0 and 2.0. As in Table 2, three different sample sizes were considered:  $n = 10, 25, 50$ . The figures in Table 4 reveal that the likelihood

TABLE 2. Null rejection rates, inference on  $\alpha$  ( $\beta = 1.0$ ).

$n$	nominal level	$\alpha = 0.1$				$\alpha = 0.5$				$\alpha = 1.0$				$\alpha = 2.0$			
		$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$
10	10	13.44	11.00	11.00	7.24	13.57	10.85	10.86	9.03	13.97	10.65	10.78	10.61	16.10	9.89	8.99	15.90
	5	7.47	5.14	5.14	3.05	7.60	5.18	5.20	3.96	7.83	5.24	5.29	5.11	9.44	5.20	4.58	9.40
	1	1.70	1.07	1.07	0.45	1.72	1.11	1.11	0.69	1.83	1.06	1.11	0.93	2.53	1.04	1.96	2.51
	0.5	0.89	0.53	0.53	0.16	0.91	0.52	0.52	0.31	0.97	0.52	0.55	0.51	1.40	0.48	0.40	1.39
25	10	10.84	10.15	10.15	8.42	10.88	10.13	10.14	9.15	11.56	10.41	10.50	10.30	11.98	10.12	10.10	11.96
	5	5.94	5.33	5.33	4.32	5.93	5.34	5.35	4.67	6.18	5.31	5.33	5.28	6.24	5.36	5.30	6.22
	1	1.43	1.18	1.18	0.83	1.46	1.15	1.15	1.02	1.46	1.04	1.04	1.04	1.54	1.07	1.06	1.54
	0.5	0.82	0.57	0.57	0.43	0.82	0.59	0.59	0.48	0.61	0.50	0.52	0.43	0.82	0.59	0.54	0.81
50	10	10.89	10.35	10.35	9.46	10.86	10.35	10.34	9.99	10.93	10.31	10.32	10.25	11.24	10.48	10.47	11.23
	5	5.54	5.21	5.21	4.58	5.53	5.20	5.19	4.81	5.40	5.03	5.05	4.86	5.65	4.95	5.04	5.64
	1	1.19	0.99	0.99	0.87	1.21	1.02	1.02	0.99	1.14	1.02	1.02	1.10	1.18	1.03	1.03	1.17
	0.5	0.70	0.56	0.56	0.51	0.71	0.55	0.55	0.58	0.68	0.61	0.61	0.61	0.60	0.49	0.51	0.60

TABLE 3. Nonnull rejection rates, inference on  $\alpha$ .

$\alpha$	nominal level: 5%				nominal level: 1%			
	$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR^*$
0.12	8.55	13.13	13.13	8.56	2.50	4.87	4.87	2.50
0.14	24.79	34.07	34.07	24.79	11.71	18.01	18.01	11.71
0.16	45.74	56.06	56.06	45.75	29.37	38.16	38.16	29.37
0.18	64.78	72.63	72.63	64.58	48.57	59.91	59.91	48.57
0.20	77.86	83.81	83.81	77.87	65.33	72.58	72.58	65.33
0.22	86.41	89.85	89.85	86.41	77.29	82.97	82.97	77.29
0.24	91.25	94.18	94.18	91.25	85.55	88.88	88.88	85.55
0.26	94.91	96.73	96.73	94.91	90.17	93.01	93.01	90.17
0.28	96.98	97.95	97.95	96.98	93.90	95.40	95.40	93.90

ratio test is liberal, that  $LR_{BN}$  is conservative, and that  $LR_{CR}$  displays very minor size distortions, the latter clearly outperforming the other tests. For instance, when  $n = 10$ ,  $\alpha = 2.0$  and at the 10% nominal level, the null rejection rates of these tests are, respectively, 13.12%, 7.94% and 10.66%.

In Table 5 we present the empirical powers of the tests of  $\mathcal{H}_0 : \beta = \beta_0$ . The values of  $\beta$  used ranged from 1.2 to 4.0. Again, the tests were performed using size-corrected critical values (obtained from the size simulations) in order to force them to have the correct size. The simulations were carried out using  $n = 10$ ,  $\alpha = 1.0$  and  $\beta = 1.0$ . The results suggest that the powers of the three tests are very similar, with a slight advantage of  $LR$ . For example, when  $\beta = 2.0$ , the nonnull rejection rates of  $LR$ ,  $LR_{CR}$  and  $LR_{BN}$  at the 5% nominal level are, respectively, 58.54%, 58.05% and 57.43%.

Figure 3 plots the relative quantile discrepancies against the corresponding asymptotic quantiles, and Figure 4 plots the relative size distortions against the corresponding nominal levels of the tests when  $\beta$  is the parameter of interest. Both figures clearly show that  $LR_{CR}$  outperforms  $LR$  and  $LR_{BN}$ .

## 6. ADDITIONAL NUMERICAL EVIDENCE: COMPARISON WITH ALTERNATIVE ESTIMATORS

We shall now compare the small sample behavior of our adjusted profile maximum likelihood estimators of  $\alpha$  to those proposed by Ng, Kundu and Balakrishnan (2003) and From and Li (2006), which are described in Section 4. The number of Monte Carlo replications is, as before, 10,000, the true values of  $\alpha$  are 0.5 and 1.0, and  $n = 10$ . The simulation results are presented in Table 6.

The figures in Table 6 show that no estimator uniformly outperforms all others in terms of both bias and mean squared error. They also show that the adjusted profile maximum likelihood estimators are competitive, since they are amongst the best performing estimators in both situations ( $\alpha = 0.5$  and  $\alpha = 1.0$ ). When  $\alpha = 0.5$ , the least biased estimator is  $\check{\alpha}_{F4}$  (relative bias: 0.66%) whereas when the true parameter value is 1.0 the estimator  $\check{\alpha}_{F3}$  has the smallest relative bias (0.26%). In both cases,  $\widehat{\alpha}_{BN}$  is the estimator with the third smallest relative bias, followed

TABLE 4. Null rejection rates, inference on  $\beta$  ( $\beta = 1.0$ ).

$n$	nominal level	$\alpha = 0.1$			$\alpha = 0.5$			$\alpha = 1.0$			$\alpha = 2.0$		
		$LR$	$LR_{CR}$	$LR_{BN}$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR$	$LR_{CR}$	$LR_{BN}$	$LR$	$LR_{CR}$	$LR_{BN}$
10	10	12.31	10.30	8.52	12.33	10.25	8.26	12.37	10.18	8.01	13.12	10.66	7.94
	5	6.61	5.35	3.95	6.62	5.30	3.85	6.75	5.18	3.65	6.95	5.27	3.65
	1	1.49	1.08	0.70	1.56	1.05	0.69	1.55	1.09	0.67	1.73	1.12	0.54
	0.5	0.91	0.59	0.35	0.89	0.60	0.33	0.87	0.59	0.28	0.97	0.53	0.25
25	10	11.10	10.38	9.67	11.19	10.33	9.60	11.15	10.37	9.52	11.27	10.52	9.53
	5	5.83	5.30	4.86	5.91	5.38	4.76	5.87	5.33	4.75	6.14	5.49	4.77
	1	1.25	1.08	0.93	1.30	1.07	0.99	1.40	1.15	0.97	1.19	1.09	0.96
	0.5	0.62	0.52	0.44	0.64	0.52	0.45	0.76	0.61	0.46	0.80	0.64	0.43
50	10	10.71	10.43	9.96	10.64	10.35	9.92	10.72	10.24	9.74	10.87	10.38	9.87
	5	5.32	5.05	4.85	5.36	5.17	4.88	5.54	5.17	4.90	5.39	5.07	4.79
	1	1.19	1.08	1.00	1.20	1.10	0.99	1.18	1.09	0.98	1.12	1.06	0.97
	0.5	0.56	0.51	0.48	0.58	0.53	0.46	0.61	0.55	0.50	0.65	0.59	0.52

TABLE 5. Nonnull rejection rates, inference on  $\beta$ .

$\beta$	nominal level: 5%			nominal level: 1%		
	$LR$	$LR_{CR}$	$LR_{BN}$	$LR$	$LR_{CR}$	$LR_{BN}$
1.2	9.17	9.07	9.06	2.06	2.06	2.07
1.6	31.67	31.32	31.06	10.32	10.18	9.94
2.0	58.54	58.05	57.43	24.56	24.21	23.50
2.4	80.11	79.72	79.13	45.06	44.07	42.93
2.8	91.05	90.93	90.49	62.15	61.08	59.56
3.2	96.80	96.53	96.42	75.86	74.73	72.68
3.6	99.05	99.01	98.85	85.86	84.74	83.10
4.0	99.61	99.55	99.45	92.34	91.29	89.75

by  $\widehat{\alpha}_{CR}$ . When  $\alpha = 0.5$ , for instance, the relative biases of these estimators are nearly four times smaller than those of the jackknife estimators and nearly 3.5 times smaller than the relative bias of the modified moments estimator.

We note that the approach proposed in this paper, namely adjusting the profile log-likelihood function prior to maximization, has a clear advantage over the alternative approaches described in Section 4: it not only improves the small sample performance of point estimators, but also improves the finite sample behavior of associated likelihood ratio tests. That is, the correction delivers improved estimation *and* testing inference in small samples.

## 7. APPLICATIONS

We shall now perform profile and adjusted profile likelihood inference using two real data sets. In both cases, we shall assume that observations are random draws from the Birnbaum–Saunders distribution.

At the outset, we consider the data provided by Birnbaum–Saunders (1969b) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data set consists of 101 observations with maximum stress per cycle 31,000 psi. Let  $\alpha$  be the parameter of interest. The profile and adjusted profile maximum likelihood estimates are  $\widehat{\alpha} = 0.17038$ ,  $\widehat{\alpha}_{CR} = 0.17125$  and  $\widehat{\alpha}_{BN} = 0.17122$ . Suppose we are interested in testing  $\mathcal{H}_0 : \alpha = 0.15$  against  $\mathcal{H}_1 : \alpha \neq 0.15$ . The test statistics based on  $\ell_p(\alpha)$ ,  $\ell_{CR}(\alpha)$  and  $\ell_{BN}(\alpha)$  are, respectively, 3.5771, 3.8421 and 3.8351, with the following corresponding  $p$ -values: 0.05858, 0.04998 and 0.05019. Since the sample size is large (101 observations), the values of the three statistics are similar. However, the resulting inference is not the same at the 5% nominal level, since the test based on  $\ell_{CR}(\alpha)$ , unlike the other two tests, yields rejection of the null hypothesis.

We shall now turn to the case where  $\beta$  is the parameter of interest. In particular, we are interested in testing  $\mathcal{H}_0 : \beta = 125$ . The test statistics are  $LR = 9.4279$ ,  $LR_{CR} = 9.3338$  and  $LR_{BN} = 9.2397$ , with the following corresponding  $p$ -values: 0.00214, 0.00225 and 0.00237. Unlike the previous inference, here the three tests



TABLE 6. Point estimation of  $\alpha$  revisited.

$\alpha = 0.5$							
estimator	mean	bias	variance	MSE	RB(%)	asymmetry	kurtosis
$\widehat{\alpha}_p$	0.4625	-0.0375	0.0121	0.0135	7.5074	1.3537	5.9906
$\widehat{\alpha}_{CR}$	0.4892	-0.0109	0.0136	0.0138	2.1695	1.4258	6.3098
$\widehat{\alpha}_{BN}$	0.4893	-0.0107	0.0137	0.0138	2.1385	1.4624	6.3117
$\widetilde{\alpha}_{MME}$	0.4625	-0.0375	0.0121	0.0135	7.5077	1.3537	5.9906
$\widetilde{\alpha}_{Ng}$	0.4749	-0.0251	0.0127	0.0133	5.0275	1.3874	6.1376
$\widetilde{\alpha}_{NgJMLE}$	0.4573	-0.0427	0.0133	0.0152	8.5384	1.3397	5.9301
$\widetilde{\alpha}_{NgJMME}$	0.4579	-0.0421	0.0133	0.0151	8.4255	1.3412	5.9367
$\check{\alpha}_{F1}$	0.4625	-0.0375	0.0121	0.0135	7.4949	1.3539	5.9913
$\check{\alpha}_{F2}$	0.4690	-0.0310	0.0166	0.0176	6.2008	1.3715	6.0678
$\check{\alpha}_{F3}$	0.5065	0.0065	0.0326	0.0326	1.3004	1.4721	6.5222
$\check{\alpha}_{F4}$	0.5033	0.0033	0.0220	0.0220	0.6551	1.4635	6.4825
$\alpha = 1.0$							
estimator	mean	bias	variance	MSE	RB(%)	asymmetry	kurtosis
$\widehat{\alpha}_p$	0.9160	-0.0840	0.0472	0.0543	8.4020	2.3784	11.8471
$\widehat{\alpha}_{CR}$	0.9752	-0.0248	0.0548	0.0554	2.4841	2.4786	12.5612
$\widehat{\alpha}_{BN}$	0.9792	-0.0208	0.0567	0.0572	2.0834	2.4852	12.6085
$\widetilde{\alpha}_{MME}$	0.9158	-0.0842	0.0471	0.0542	8.4173	2.3782	11.8453
$\widetilde{\alpha}_{Ng}$	0.9404	-0.0596	0.0497	0.0533	5.9615	2.4207	12.1452
$\widetilde{\alpha}_{NgJMLE}$	0.9050	-0.0950	0.0523	0.0613	9.4959	2.3591	11.7120
$\widetilde{\alpha}_{NgJMME}$	0.9061	-0.0939	0.0522	0.0611	9.3931	2.3609	11.7247
$\check{\alpha}_{F1}$	0.9168	-0.0832	0.0475	0.0544	8.3177	2.3799	11.8575
$\check{\alpha}_{F2}$	0.8916	-0.1084	0.0763	0.0881	10.8379	2.3350	11.5450
$\check{\alpha}_{F3}$	0.9974	-0.0026	0.1279	0.1279	0.2598	2.5144	12.8217
$\check{\alpha}_{F4}$	1.0039	0.0039	0.0906	0.0906	0.3915	2.5247	12.8972

yield the same conclusion: the null hypothesis is rejected at the 1% nominal level. The (profile and adjusted profile) maximum likelihood estimate of  $\beta$  is 131.8188.

The second application we consider uses data provided by McCool (1974). The data describe the lifetime, in hours, of 10 sustainers of a certain type. They were used by Cohen, Whitten and Ding (1984) as an illustration of the three-parameter Weibull distribution. The profile and adjusted profile maximum likelihood estimates of  $\alpha$  are  $\hat{\alpha} = 0.2825$ ,  $\hat{\alpha}_{CR} = 0.2989$  and  $\hat{\alpha}_{BN} = 0.2973$ . Consider the test of  $\mathcal{H}_0 : \alpha = 0.21$  against  $\mathcal{H}_1 : \alpha \neq 0.21$ . The test statistics are  $LR = 2.1646$ ,  $LR_{CR} = 2.8438$  and  $LR_{BN} = 2.7963$ , with the following corresponding  $p$ -values: 0.1412, 0.0917 and 0.0945. Therefore, the two adjusted profile likelihood ratio tests ( $LR_{CR}$  and  $LR_{BN}$ ) reject the null hypothesis at the 10% nominal level, unlike the likelihood ratio test ( $LR$ ). Thus, the unadjusted and adjusted tests yield different conclusions.

Now let  $\beta$  be the parameter of interest. Its maximum likelihood estimate is  $\hat{\beta} = 212.05$ . Suppose we wish to test  $\mathcal{H}_0 : \beta = 180$  against  $\mathcal{H}_1 : \beta \neq 180$ . The test statistics are  $LR = 2.9417$ ,  $LR_{CR} = 2.6415$  and  $LR_{BN} = 2.3414$ ; the respective  $p$ -values are 0.0863, 0.1041 and 0.1260. Again, the use of adjustments to the profile likelihood function makes a difference: unlike the usual likelihood ratio test, the adjusted likelihood tests reject the null hypothesis at the 10% nominal level.

## 8. CONCLUDING REMARKS

We considered the issue of performing inference on the parameters that index the Birnbaum–Saunders distribution. More specifically, we have focused on the situation where one wishes to make inference on one of the parameters, the other parameter being of nuisance fashion. Using the results in Cox and Reid (1987) and in Barndorff–Nielsen (1983), we derived two adjustments that can be applied to the profile likelihood function so as to deliver improved inference. Approximations due to Severini (1998, 1999) were used in order to obtain one of such adjustments. Monte Carlo simulation results have shown that the adjusted estimators and tests — i.e., estimators and tests based on the adjusted profile likelihood functions — can deliver more accurate inference than that carried out using the usual maximum likelihood estimator and the standard likelihood ratio test in small samples. In particular, the adjusted estimators displayed smaller biases and the adjusted tests, smaller size distortions. For instance, we reported Monte Carlo simulation results in which the usual likelihood ratio test displayed null rejection rate of nearly 9.5% at the 5% nominal level whereas the sizes of our two adjusted tests were 5.2% and 4.6%, and in which the relative bias of our two estimators were approximately four times smaller than that of the maximum likelihood estimator. The adjusted profile likelihood tests have also outperformed the Bartlett-corrected likelihood ratio test. We recommend the use of the adjusted profile likelihood inference developed in this paper to practitioners who wish to model reliability data using the Birnbaum–Saunders model. In particular, we recommend the use of the Cox–Reid adjusted profile likelihood function, since it yielded the most reliable (hypothesis testing) inference when the parameter of interest was  $\beta$  and it was competitive with the

inference obtained using the Barndorff–Nielsen modified profile likelihood function when  $\alpha$  was the parameter of interest. In future research, we shall obtain adjustments to Birnbaum–Saunders profile likelihoods under data censoring.

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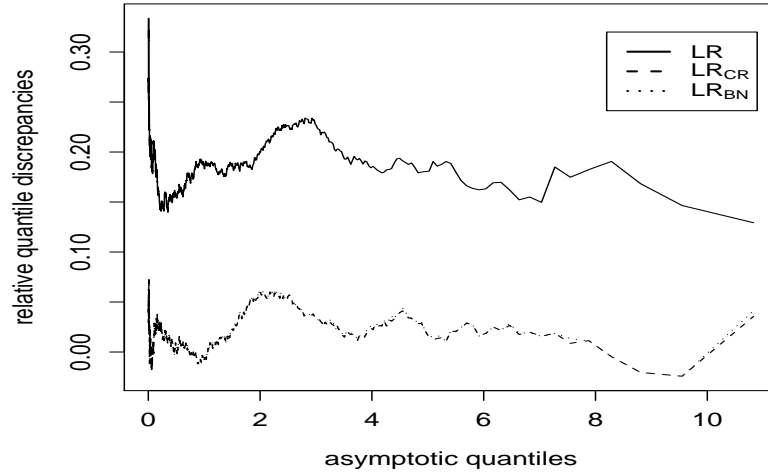
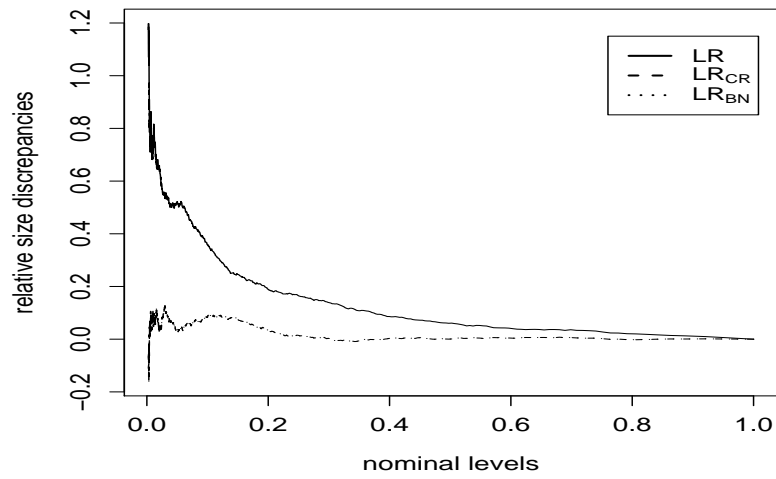
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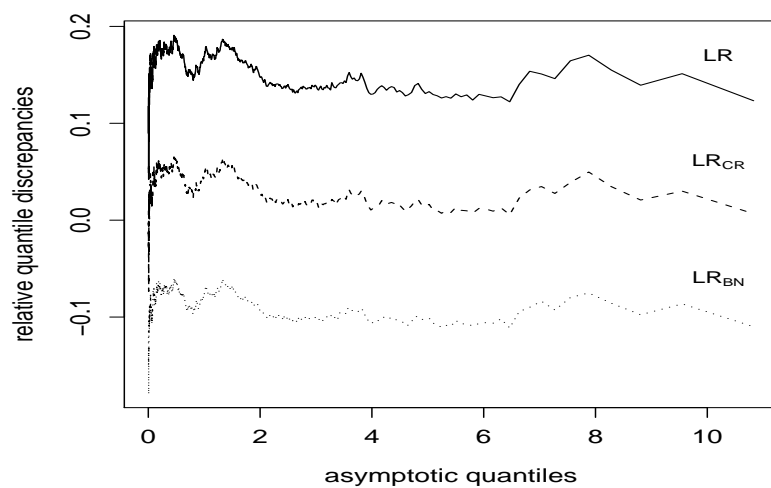
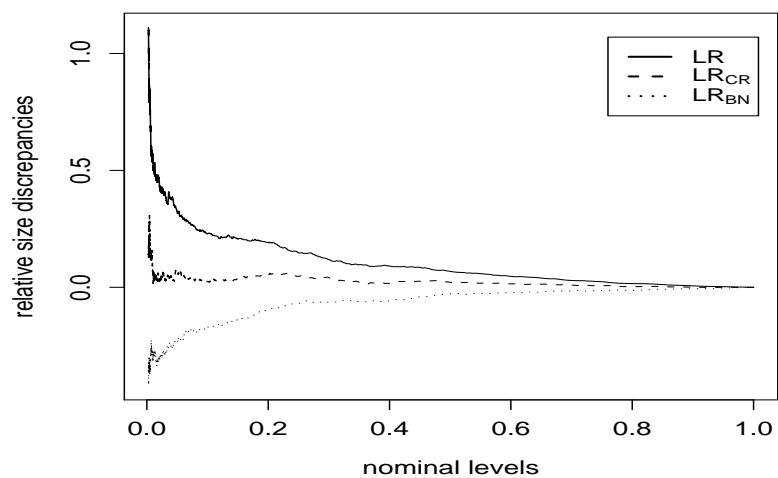
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FIGURE 1. Relative quantile discrepancy plot, inference on  $\alpha$ .FIGURE 2. Relative size distortion plot, inference on  $\alpha$ .

FIGURE 3. Relative quantile discrepancy plot, inference on  $\beta$ .FIGURE 4. Relative size distortion plot, inference on  $\beta$ .